High Energy Commutators in Particle, String and Membrane Theories

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February 1, 2008

Abstract

We study relativistic particle, string and membrane theories as defining field theories containing gravity in (0+1), (1+1) and (2+1) space-time dimensions, respectively. We show how an off shell invariance of the massless particle action allows the construction of an extension of the conformal algebra and induces a transition to a non-commutative space-time geometry. This non-commutative geometry is found to be preserved in the space-time supersymmetric massless particle theory. It is then shown how the basic bosonic commutators we found for the massless particle may also be encountered in the tensionless limit of string and membrane theories. Finally we speculate on how the non-locality introduced by these commutators could be used to construct a covariant Newtonian gravitational field theory.

1 Introduction

Since some time ago there has been some interest in explaining a very small measured value for the cosmological constant. It is believed that this tinny value is a reflection of the presence of large amounts of dark matter in the universe [1,2]. Because the measured value is positive, dark matter should be contributing with positive gravitational potential energy as a consequence of its interaction with usual matter. This implies that dark matter should repel usual matter. We may add to this situation the observation [3], using high resolution measurements of the cosmic microwave background radiation, that the universe is effectively flat. On theoretical grounds there is an interesting result obtained by Siegel [4], based in scalar particle theory, that there are stringy corrections to gravitation and that these corrections make the gravitational force less attractive at very short distances. This work is an attempt to sketch a mathematical picture in which all the above mentioned subjects would occupy mathematically natural and consistent positions. The central mathematical concept in this attempt will be non-commutativity of space-time coordinates

Non-commutativity is the central mathematical concept expressing uncertainty in quantum mechanics, where it applies to any pair of conjugate variables. But there are other situations where non-commutativity may manifest itself. For instance, in the presence of a magnetic field even momenta fail to mutually commute. The modern trend is to believe that one might postulate non-commutativity for a number of reasons. One of these is the belief that in quantum theories including gravity, space-time must change its nature at the Planck scale. This is because quantum gravity has an uncertainty principle which prevents one from measuring positions to better accuracies than the Planck length. The momentum and energy required to make such a measurement will itself modify the space-time geometry at this scale [5]. This effect could then be modeled by a non-vanishing commutation relation between the space-time coordinates.

For most theories, postulating an uncertainty relation between space-time coordinates will lead to a non-local theory and this may conflict with Lorentz invariance. However, there is a remarkable exception to this situation: string theory. String theory is not local in any sense we now understand and indeed it has more than one parameter characterizing this non-locality. One of these is the string length l_s, the average size of a string. As we will see in this work, another useful parameter is the string tension T , the string analogue of the particle's mass. The string tension defines an energy scale in string theory and the vanishing tension limit is expected to describe the string dynamics at the Planck scale [6].

Historically, the first use of non-commutative geometry in string theory was in Witten's formulation of open string field theory [7]. Witten's formulation of non-commutative string field theory is based in the introduction of a star product which is defined as the overlap of half of two strings. The central idea in this formulation is to decompose the string coordinates $x^\mu[\sigma]$ as the direct sum $x^\mu[\sigma] = l^\mu[\sigma] \oplus r^\mu[\sigma]$, where $l^\mu[\sigma]$ and $r^\mu[\sigma]$ are respectively left and right coordinates of the string, defined relative to the midpoint at $\sigma = \pi/2$. If we now consider the fields of two strings, say $A(l_1^\mu[\sigma], r_1^\mu[\sigma])$ and $B(l_2^\mu[\sigma], r_2^\mu[\sigma])$, then Witten's star product gives

$$A(l_1^\mu[\sigma], r_1^\mu[\sigma]) \star B(l_2^\mu[\sigma], r_2^\mu[\sigma]) = C(l_1^\mu[\sigma], r_2^\mu[\sigma]) \tag{1.1}$$

where the field $C(l_1^\mu[\sigma], r_2^\mu[\sigma])$ is given by the functional integral

$$C(l_1^\mu[\sigma], r_2^\mu[\sigma]) = \int [dz] A(l_1^\mu[\sigma], z^\mu[\sigma]) B(z^\mu[\sigma], r_2^\mu[\sigma]) \tag{1.2}$$

and $z^\mu[\sigma] = r_1^\mu[\sigma] = l_2^\mu[\sigma]$ corresponds to the overlap of half of the first string with half of the second string.

Witten's formulation of open bosonic string field theory has experienced a rebirth through new physical insights and technical advances. In particular, it was shown by Bars [8] that Witten's star product, originally defined as a path integral that saws two strings into a third one, is equivalent to the Moyal star product [9], which is the central mathematical concept in the familiar formulation of non-commutative geometry. But this equivalence is rather subtle. As

shown in [8], Witten's star product is equivalent to the usual Moyal star product in the relativistic phase space of only the even modes of oscillation of the string. This is because the mapping of the Witten $\star$ to the Moyal $\star$ necessarily requires that the Fourier space for the odd positions be named as the even momenta since (x_e, p_e) are canonical under the Moyal star-product. These developments, and many others that have recently appeared in the literature, led to the belief that non-commutative geometry naturally arises in string field theory.

However, because it was defined for applications in traditional quantum mechanics, the Moyal star product leaves unchanged the Heisenberg commutation relations between the canonical operators. In the usual formulations of field theory on non-commutative spaces, the position coordinates satisfy commutation relations of the form [10,11]

$$[q^\mu, q^\nu] = i\theta^{\mu\nu} \tag{1.3}$$

where $\theta^{\mu\nu}$ is a constant anti-symmetric tensor. The effect of such a modification is reflected in the momentum space vertices of the theory by factors of the form [12]

$$\exp\{i\theta^{\mu\nu} p_\mu q_\nu\} \tag{1.4}$$

The first quantized string in the light cone gauge, as a perturbative non-commu tative theory based in a commutation relation such as (1.3), was studied in [12]. It was found by the authors in [12] that, apart from trivial factors, the structure of the non-commutative theory is identical to the structure of the commutative one. It thus seems that nothing substantially new is introduced in string theory by the use of a Moyal star-product structure, or by postulating another star-product structure based on commutators such as (1.3).

In this work we give a small contribution to this subject by showing how we may use the special-relativistic ortogonality between velocity and acceleration to induce the appearance of a new invariance of the massless relativistic particle action. This invariance may then be used to generate transitions to new space-time coordinates which, in the simplest case, satisfy commutator relations of the form

$$[x_\mu, x_\nu] \sim iM^*_{\mu\nu} \tag{1.5}$$

where $M^*_{\mu\nu}$ is an off shell anti-symmetric operator that generates space-time rotations. This commutation relation is part of a modified Heisenberg commutator structure which reduces to the usual canonical commutators when the mass shell condition is imposed. Most part of this work is the explicit verification that the basic commutator structure we found for the massless particle may also be encountered in string and membrane theory. As pointed out in [11], any finite-dimensional deformed Poisson structure can be canonically quantized. It may be then interesting to investigate what quantum effects we may find in particle, string and membrane theories if we construct the corresponding non-commutative theories based on commutators such as (1.5) instead of (1.3).

An important point to our considerations here is that space-time non-commutativity introduces an associated non-locality [10]. In non-commutative gauge

theories based on the commutator (1.3) this non-locality allows the interpretation of the gauge particles as dipoles carrying opposite charges of the corresponding gauge theory [10, 12]. In the context of string theory, it was shown [13] that an open string with both ends on a (D2-D0)-brane system in which a magnetic field is defined looks like an electric dipole if a background electric field is added. The dipole-background interaction gives the flux modifications needed for the Born-Infeld action on the non-commutative torus and results in a Hamiltonian that depends on two integers, n and m, whose ratio $\frac{m}{n}$ gives the density of D0-branes distributed on the D2-brane [13].

In this work we study relativistic particle, string and membrane theories as gauge systems in which the gauge invariance is general covariance. We interpret these relativistic objects as defining field theories containing gravity in (0+1), (1+1) and (2+1) space-time dimensions, respectively. The paper is organized as follows: For the task of clarity, in section two we briefly review the concept of a space-time conformal vector field and the associated conformal algebra. In section three we discuss particle theory and show how we may induce an off shell invariance of the massless particle action and how this invariance allows the construction of an off shell extension of the conformal algebra and how it permits a transition to non-commutative space-time coordinates. In section four we show that the bosonic commutator structure we found for the massless particle is preserved in the space-time supersymmetric extension of the theory. Section five deals with bosonic strings. We present an alternative string action that in principle does not require a critical dimension and that allows a smooth transition to the tensionless limit of string theory. We show how this alternative string action may be gauge-fixed to yield a non-commutative theory at the unitary tension value and to give the extension of the particle commutators in the vanishing tension limit. Section six extends these results to superstring theory and section seven briefly describes the same situation in relativistic membrane theory. We present our conclusions in section eight, where we also speculate on the possibility of using the non-locality introduced by our commutation relations to provide the conceptual basis for a covariant Newtonian gravitational field theory.

2 Conformal Vector Fields

Consider the Euclidean flat space-time vector field

$$\hat{R}(\epsilon) = \epsilon^{\mu}\partial_{\mu} \tag{2.1}$$

such that

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \delta_{\mu\nu}\partial.\epsilon \tag{2.2}$$

The vector field $\hat{R}(\epsilon)$ gives rise to the coordinate transformation

$$\delta x^{\mu} = \hat{R}(\epsilon)x^{\mu} = \epsilon^{\mu} \tag{2.3}$$

The vector field (2.1) is known as the Killing vector field and ϵ^μ is known as the Killing vector. One can show that the most general solution for equation (2.2) in a four-dimensional space-time is

$$\epsilon^\mu = \delta x^\mu = a^\mu + \omega^{\mu\nu} x_\nu + \alpha x^\mu + (2x^\mu x^\nu - \delta^{\mu\nu} x^2) b_\nu \tag{2.4}$$

The vector field $\hat{R}(\epsilon)$ for the solution (2.4) can then be written as

$$\hat{R}(\epsilon) = a^\mu P_\mu - \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu} + \alpha D + b^\mu K_\mu \tag{2.5}$$

where

$$P_\mu = \partial_\mu \tag{2.6}$$

$$M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \tag{2.7}$$

$$D = x^\mu \partial_\mu \tag{2.8}$$

$$K_\mu = (2x_\mu x^\nu - \delta_\mu^\nu x^2)\partial_\nu \tag{2.9}$$

P_μ generates translations, $M_{\mu\nu}$ generates rotations, D generates dilatations and K_μ generates conformal transformations in space-time. The generators of the vector field $\hat{R}(\epsilon)$ obey the commutator algebra

$$[P_\mu, P_\nu] = 0 \tag{2.10a}$$

$$[P_\mu, M_{\nu\lambda}] = \delta_{\mu\nu} P_\lambda - \delta_{\mu\lambda} P_\nu \tag{2.10b}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \delta_{\nu\lambda} M_{\mu\rho} + \delta_{\mu\rho} M_{\nu\lambda} - \delta_{\nu\rho} M_{\mu\lambda} - \delta_{\mu\lambda} M_{\nu\rho} \tag{2.10c}$$

$$[D, D] = 0 \tag{2.10d}$$

$$[D, P_\mu] = -P_\mu \tag{2.10e}$$

$$[D, M_{\mu\nu}] = 0 \tag{2.10f}$$

$$[D, K_\mu] = K_\mu \tag{2.10g}$$

$$[P_\mu, K_\nu] = 2(\delta_{\mu\nu} D - M_{\mu\nu}) \tag{2.10h}$$

$$[M_{\mu\nu}, K_\lambda] = \delta_{\nu\lambda} K_\mu - \delta_{\lambda\mu} K_\nu \tag{2.10i}$$

$$[K_\mu, K_\nu] = 0 \tag{2.10j}$$

The commutator algebra (2.10) is the conformal space-time algebra in four dimensions. Notice that the commutators (2.10a-2.10c) correspond to the Poincaré algebra. The Poincaré algebra is a sub-algebra of the conformal algebra. Let us now see how we can extend the conformal algebra, and how this extended algebra is related to a non-commutative space-time geometry.

3 Relativistic Particles

A relativistic particle describes is space-time a one-parameter trajectory $x^\mu(\tau)$. The dynamics of the particle must be independent of the parameter choice. A possible form of the action is the one proportional to the arc length traveled by the particle and given by

$$S = -m \int ds = -m \int d\tau \sqrt{-\dot{x}^2} \tag{3.1}$$

In this work we take the parameter τ to be the particle's proper time, m is the particle's mass and $ds^2 = -\delta_{\mu\nu} dx^\mu dx^\nu$. A dot denotes derivatives with respect to τ and we use units in which $\hbar = c = 1$.

Action (3.1) is obviously inadequate to study the massless limit of the theory and so we must find an alternative action. Such an action can be easily computed by treating the relativistic particle as a constrained system. In the transition to the Hamiltonian formalism action (3.1) gives the canonical momentum

$$p_\mu = \frac{m}{\sqrt{-\dot{x}^2}} \dot{x}_\mu \tag{3.2}$$

and this momentum gives rise to the primary constraint

$$\phi = \frac{1}{2}(p^2 + m^2) = 0 \tag{3.3}$$

We follow Dirac's [14] convention that a constraint is set equal to zero only after all calculations have been performed. The canonical Hamiltonian corresponding to action (3.1), $H = p.\dot{x} - L$, identically vanishes. This is a characteristic feature of reparametrization-invariant systems. Dirac's Hamiltonian for the relativistic particle is then

$$H_D = H + \lambda\phi = \frac{1}{2}\lambda(p^2 + m^2) \tag{3.4}$$

where $\lambda(\tau)$ is a Lagrange multiplier. The Lagrangian that corresponds to (3.4) is

$$\begin{aligned} L &= p.\dot{x} - H_D \\ &= p.\dot{x} - \frac{1}{2}\lambda(p^2 + m^2) \end{aligned} \tag{3.5}$$

Solving the equation of motion for p_μ that follows from (3.5) and inserting the result back in it, we obtain the particle action

$$S = \int d\tau (\frac{1}{2}\lambda^{-1}\dot{x}^2 - \frac{1}{2}\lambda m^2) \tag{3.6}$$

The great advantage of action (3.6) is that it has a smooth transition to the $m = 0$ limit.

Action (3.6) is invariant under the Poincaré transformations

$$\delta x^\mu = a^\mu + \omega^\mu_\nu x^\nu \tag{3.7a}$$

$$\delta\lambda = 0 \tag{3.7b}$$

Invariance of action (3.6) under transformation (3.7a) implies that we can construct a space-time vector field corresponding to the first two generators in the right of equation (2.5). These generators realize the Poincaré algebra (2.10a-2.10c).

Now we make a transition to the massless limit. This limit is described by the action

$$S = \frac{1}{2}\int d\tau \lambda^{-1}\dot{x}^2 \tag{3.8}$$

The massless limit is usually considered to describe the high energy limit of relativistic particle theory. The canonical momentum conjugate to x^μ is

$$p_\mu = \frac{1}{\lambda}\dot{x}_\mu \tag{3.9}$$

The canonical momentum conjugate to λ identically vanishes and this is a primary constraint, $p_\lambda = 0$. Constructing the canonical Hamiltonian, and requiring the stability of this constraint, we are led to the mass shell condition

$$\phi = \frac{1}{2}p^2 = 0 \tag{3.10}$$

Let us now study which space-time symmetries are present in this limit. Being the $m = 0$ limit of (3.6), action (3.8) is also invariant under the Poincaré transformation (3.7). The massless action (3.8) however has a larger set of space-time invariances. It is also invariant under the scale transformation

$$\delta x^\mu = \alpha x^\mu \tag{3.11a}$$

$$\delta\lambda = 2\alpha\lambda \tag{3.11b}$$

where α is a constant, and under the conformal transformation

$$\delta x^\mu = (2x^\mu x^\nu - \delta^{\mu\nu}x^2)b_\nu \tag{3.12a}$$

$$\delta\lambda = 4\lambda x.b \tag{3.12b}$$

where b_μ is a constant vector. Invariance of action (3.8) under transformations (3.7a), (3.11a) and (3.12a) then implies that the full conformal field (2.5) can be defined in the massless sector of the theory.

It is convenient at this point to relax the mass shell condition (3.10) because the massless particle will be off mass shell in the presence of interactions [15]. We may now use the fact that we are dealing with a special-relativistic system. Special relativity has the characteristic kinematical feature that the relativistic velocity is always ortogonal to the relativistic acceleration (see, for instance, [16]). While the mathematics involved at this point is very simple, the physical implications of this ortogonality condition are rather unexplored. In this work we make an attempt to partially clarify these physical implications in the context of massless relativistic particle theory. We start by adopting the point of view that special relativity automatically implies the ortogonality condition between the velocity and the acceleration. Reasoning in this way, we find that we may use this ortogonality to induce the presence of a new invariance of the massless particle action. Then, as a consequence of the fact that action (3.8) is a special-relativistic action, it is invariant under the transformation

$$x^\mu \rightarrow \tilde{x}^\mu = \exp\{\beta(\dot{x}^2)\}x^\mu \tag{3.13a}$$

$$\lambda \rightarrow \exp\{2\beta(\dot{x}^2)\}\lambda \tag{3.13b}$$

where β is an arbitrary function of $\dot{x}^2$. We emphasize that although the ortogonality condition must be used to get the invariance of action (3.8) under transformation (3.13), this condition is not an external ingredient in the theory. In fact, the ortogonality between the relativistic velocity and the acceleration is an unavoidable condition here, it is an imposition of special relativity. We are just using this aspect of special relativity to get an invariant action.

Now, invariance of the massless action under transformations (3.13) means that infinitesimally we can define the scale transformation $\delta x^\mu = \alpha\beta(\dot{x}^2)x^\mu$, where α is the same constant that appears in equations (2.4) and (2.5). These transformations then lead to the existence of a new type of dilatations. These new dilatations manifest themselves in the fact that the vector field D of equation (2.8) can be changed according to

$$D = x^\mu\partial_\mu \rightarrow D^* = x^\mu\partial_\mu + \beta(\dot{x}^2)x^\mu\partial_\mu = D + \beta D \tag{3.14}$$

In fact, because all vector fields in equation (2.5) involve partial derivatives with respect to x^μ and β is a function of $\dot{x}^\mu$ only, we can also introduce the generators

$$P^*_\mu = P_\mu + \beta P_\mu \tag{3.15}$$

$$M^*_{\mu\nu} = M_{\mu\nu} + \beta M_{\mu\nu} \tag{3.16}$$

$$K_\mu^* = K_\mu + \beta K_\mu \tag{3.17}$$

and define the new space-time vector field

$$V_0^* = a^\mu P_\mu^* - \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu}^* + \alpha D^* + b^\mu K_\mu^* \tag{3.18}$$

The generators of this vector field obey the algebra

$$[P_\mu^*, P_\nu^*] = 0 \tag{3.19a}$$

$$[P_\mu^*, M_{\nu\lambda}^*] = (\delta_{\mu\nu} P_\lambda^* - \delta_{\mu\lambda} P_\nu^*) + \beta(\delta_{\mu\nu} P_\lambda^* - \delta_{\mu\lambda} P_\nu^*) \tag{3.19b}$$

$$[M_{\mu\nu}^*, M_{\lambda\rho}^*] = (\delta_{\nu\lambda} M_{\mu\rho}^* + \delta_{\mu\rho} M_{\nu\lambda}^* - \delta_{\nu\rho} M_{\mu\lambda}^* - \delta_{\mu\lambda} M_{\nu\rho}^*)$$

$$+\beta(\delta_{\nu\lambda} M_{\mu\rho}^* + \delta_{\mu\rho} M_{\nu\lambda}^* - \delta_{\nu\rho} M_{\mu\lambda}^* - \delta_{\mu\lambda} M_{\nu\rho}^*) \tag{3.19c}$$

$$[D^*, D^*] = 0 \tag{3.19d}$$

$$[D^*, P_\mu^*] = -P_\mu^* - \beta P_\mu^* \tag{3.19e}$$

$$[D^*, M_{\mu\nu}^*] = 0 \tag{3.19f}$$

$$[D^*, K_\mu^*] = K_\mu^* + \beta K_\mu^* \tag{3.19g}$$

$$[P_\mu^*, K_\nu^*] = 2(\delta_{\mu\nu} D^* - M_{\mu\nu}^*) + 2\beta(\delta_{\mu\nu} D^* - M_{\mu\nu}^*) \tag{3.19h}$$

$$[M_{\mu\nu}^*, K_\lambda^*] = (\delta_{\lambda\nu} K_\mu^* - \delta_{\lambda\mu} K_\nu^*) + \beta(\delta_{\lambda\nu} K_\mu^* - \delta_{\lambda\mu} K_\nu^*) \tag{3.19i}$$

$$[K_\mu^*, K_\nu^*] = 0 \tag{3.19j}$$

Notice that the vanishing brackets of the conformal algebra (2.10) are preserved as vanishing in the above algebra, but the non-vanishing brackets of the conformal algebra now have linear and quadratic contributions from the arbitrary function $\beta(\dot{x}^2)$. Algebra (3.19) is an off shell extension of the conformal algebra (2.10).

Now consider the commutator structure induced by transformation (3.13a). We assume the usual commutation relations between the canonical variables, $[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0$, $[x_\mu, p_\nu] = i\delta_{\mu\nu}$. Taking $\beta(\dot{x}^2) = \beta(\lambda^2 p^2)$ in transformation (3.13a) and transforming the p_μ in the same manner as the x_μ, we find that the new transformed canonical variables $(\tilde{x}_\mu, \tilde{p}_\mu)$ obey the commutators

$$[\tilde{p}_\mu, \tilde{p}_\nu] = 0 \tag{3.20}$$

$$[\tilde{x}_\mu, \tilde{p}_\nu] = i\delta_{\mu\nu}(1+\beta)^2 + (1+\beta)[x_\mu, \beta]p_\nu \tag{3.21}$$

$$[\tilde{x}_\mu, \tilde{x}_\nu] = (1+\beta)\{x_\mu[\beta, x_\nu] - x_\nu[\beta, x_\mu]\} \tag{3.22}$$

written in terms of the old canonical variables. These commutators obey the non trivial Jacobi identities $(\tilde{x}_\mu, \tilde{x}_\nu, \tilde{x}_\lambda) = 0$ and $(\tilde{x}_\mu, \tilde{x}_\nu, \tilde{p}_\lambda) = 0$. They also reduce to the usual canonical commutators when $\beta(\lambda^2 p^2) = 0$.

It may be questioned here, on the basis of the definition (3.9) of the classical canonical momentum, if the quantum momentum should not transform in the inverse way of the x^μ under (3.13), as is the case for the classical momemtum. This is an interesting point in the mathematical developments here. In fact, the identification of the correct quantum canonical momentum is a typical problem in field theories defined over a non-commutative space-time [24]. An example is non-commutative quantum string field theory, where the momentum space Fourier transform of the odd vibrational position modes behave as the quantum canonical momenta conjugate to the even vibrational position modes [8]. In the context of this work, transforming the quantum momenta as the classical ones implies substituting the commutator (3.21) by the commutator

$$[\tilde{x}_\mu, \tilde{p}_\nu] = (1+\beta)\{i\delta_{\mu\nu}(1-\beta) - [x_\mu, \beta]p_\nu\}$$

This commutator also reduces to the usual Heisenberg commutator when $\beta(\lambda^2 p^2) = 0$. However, the Jacobi identity $(\tilde{x}_\mu, \tilde{x}_\nu, \tilde{p}_\lambda) = 0$, based on commutator (3.22) and on the above commutator does not seem to close. If this is the case, the classical and quantum canonical momenta of the massless particle on a non-commutative space-time are related by a scale transformation of the type (3.13). That is, $p_q = \exp\{2\beta\}p_c$.

The simplest example of non-commutative space-time geometry induced by transformation (3.13) is the case when $\beta(\lambda^2 p^2) = \lambda^2 p^2$. The new positions then satisfy

$$[\tilde{x}_\mu, \tilde{x}_\nu] = -2i\lambda^2 M^*_{\mu\nu} \tag{3.23}$$

where $M^*_{\mu\nu}$ is the extended off shell operator of Lorentz rotations given by equation (3.16). The commutator (3.23) satisfies

$$\int \mathbf{Tr}[\tilde{x}_\mu, \tilde{x}_\nu] = 0 \tag{3.24}$$

as is the case for a general non-commutative algebra [10]. From equation (3.23) we may say that there exists an inertial frame in which the uncertainty introduced by two simultaneous position measurements is an off shell rotation in space-time. This appears to confirm the observation in [25] that the non-commutative space-time is to be interpreted as having an internal angular momentum throughout.

We may now consider the question of if we can find a mathematical structure that would behave like a dipole in the context of massless particle theory. Such

a mathematical structure may be constructed by considering the most general relativistic particle of mass m that special relativity allows us to construct. This corresponds to a particle with an internal structure formed of positive and negative multiples of a fundamental mass μ , that is,

$$m = (n_+ + n_-)\mu$$

with $n_+ = 1, 2, ...$ and $n_- = -1, -2, ...$ The values $n_+ = n_- = 0$ are not allowed here because we interpret the fundamental mass μ as a reflection of the existence of a fundamental length. The reader should recall at this point that special relativity allows the existence of negative masses trough the equation

$$E^2 = p^2 + m^2$$

(in our system of units). We may then interpret the non-locality in space-time introduced by the commutator (3.23) as an indication that the massless relativistic particle is actually a gravitational dipole with $n_+ = 1$ and $n_- = -1$.

4 Superparticles

In this section we consider the case of the massless superparticle. The bosonic massless action (3.8) has the supersymmetric extension [17]

$$S = \frac{1}{2}\int d\tau \lambda^{-1}(\dot{x}^\mu - i\bar{\theta}\Gamma^\mu\dot{\theta})^2 \tag{4.1}$$

where θ_α is a space-time spinor and $\Gamma^\mu_{\alpha\beta}$ are Dirac matrices. Action (4.1) is invariant under the global supersymmetry transformation

$$\delta x^\mu = i\bar{\epsilon}\Gamma^\mu\theta \tag{4.2a}$$

$$\delta\theta = \epsilon \tag{4.2b}$$

$$\delta\lambda = 0 \tag{4.2c}$$

where ϵ is an infinitesimal constant Grassmann parameter. The canonical momenta that follow from action (4.1) are

$$p_\lambda = 0 \tag{4.3}$$

$$p_\mu = \frac{1}{\lambda}(\dot{x}_\mu - i\bar{\theta}\Gamma_\mu\dot{\theta}) \tag{4.4}$$

$$\pi_\alpha = i(p.\Gamma\theta)_\alpha \tag{4.5}$$

Equation (4.5) leads to the fermionic constraint $\pi_\alpha - i(p.\Gamma\theta)_\alpha = 0$. As in the bosonic case, requiring the stability of constraint (4.3), we are led to the constraint

$$\phi = \frac{1}{2}p^2 = 0 \tag{4.6}$$

which is the mass shell condition for the superparticle. It is convenient to introduce the variable $Z_0^\mu = \dot{x}^\mu - i\bar{\theta}\Gamma^\mu\dot{\theta}$. This variable is invariant under transformation (4.2) all by itself and so all manipulations involving Z_0^μ will automatically be supersymmetric invariant.

If we extend the calculations in [16] to the case of the massless superparticle, we will find that the relation

$$Z_0.\frac{dZ_0}{d\tau} = 0 \tag{4.7}$$

must hold. Equation (4.7) is the supersymmetric extension of the special-relativistic condition of ortogonality between velocity and acceleration. Again, this is an imposition of the supersymmetric relativistic theory, and not an artificially introduced external ingredient. Relaxing again the mass shell condition, we find that action (4.1) is invariant under the transformation

$$x^\mu \to \tilde{x}^\mu = \exp\{\beta(Z_0^2)\}x^\mu \tag{4.8a}$$

$$\theta_\alpha \to \tilde{\theta}_\alpha = \exp\{\frac{1}{2}\beta(Z_0^2)\}\theta_\alpha \tag{4.8b}$$

$$\lambda \to \exp\{2\beta(Z_0^2)\}\lambda \tag{4.8c}$$

where β is now an arbitrary function of Z_0^2. In the canonical formalism $\beta(Z_0^2) = \beta(\lambda^2 p^2)$. Since the bosonic momentum p^μ commutes with the fermionic canonical variables θ_α and π_α , this leaves invariant the anti-commutation relations between the fermi-onic variables but change the bosonic ones in the same way as (3.20-3.22). The commutator structure we found for the bosonic massless particle is thus preserved in the supersymmetric massless theory.

5 Relativistic strings

Strings are higher-dimensional extensions of the particle concept. As a consequence of its evolution, the string traces out a world sheet in space-time. In the form originally advocated by Nambu [18] and Goto [19], the action for a string is simply proportional to the area of its world sheet. Mathematically, one formula for the area of a sheet embedded in space-time is

$$S = -T\int d\tau d\sigma\sqrt{-g} \tag{5.1}$$

where

$$g = \det g_{ab} \tag{5.2}$$

$$g_{ab} = \delta_{\mu\nu}\partial_a x^\mu \partial_b x^\nu \tag{5.3}$$

in which $x^\mu = x^\mu(\tau, \sigma)$, $a = 0, 1$ and primes will denote derivatives with respect to σ. T is a constant of proportionality required to make the action dimensionless. It must have dimension of $(length)^{-2}$ or $(mass)^2$ and it can be shown [17] that T is actually the tension in the string. The string tension defines an energy scale in string theory, and the high energy limit of the theory corresponds to a vanishing tension value [6]. This limit is expected to describe the string dynamics at the Planck length [6].

It is difficult to work with action (5.1) because it is highly non-linear. An equivalent, but more convenient form of the action can be written if we introduce a new variable h_{ab}, which will be a metric tensor for the string world sheet geometry. This more convenient action is [20]

$$S = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h} h^{ab} \delta_{\mu\nu}\partial_a x^\mu \partial_b x^\nu \tag{5.4}$$

Action (5.4) is the standard form for coupling D massless scalar fields x^μ to (1+1)-dimensional gravity [21]. Since the derivatives of h_{ab} do not appear in action (5.4) its equation of motion is a constraint and h_{ab} can be integrated out, giving back action (5.1).

Actions (5.1) and (5.4) are invariant under general transformations of the world sheet coordinates, $\tau \to \tau'(\tau, \sigma)$, $\sigma \to \sigma'(\tau, \sigma)$. This reparametrization invariance is essential for solving the classical equation of motion for x^μ that follows from action (5.4). The symmetric tensor h_{ab} has three independent components, and by a suitable choice of new parameters, τ' and σ', one can gauge away two of these components. This leaves only one independent component. However, there is one more local two-dimensional symmetry of the string action (5.4). There is a local Weyl scaling of the metric

$$h_{ab} \to \Lambda(\tau, \sigma) h_{ab} \tag{5.5}$$

which leaves the factor $\sqrt{-h}h^{ab}$ invariant. The reparametrization invariance together with the Weyl scaling can then be used to gauge away all the three independent components of h_{ab} by imposing the conformal gauge $h_{ab} = \eta_{ab}$, where η_{ab} is the flat two-dimensional metric. In this gauge action (5.4) becomes

$$S = -\frac{T}{2}\int d\tau d\sigma \eta^{ab} \delta_{\mu\nu}\partial_a x^\mu \partial_b x^\nu \tag{5.6}$$

The equation of motion derived from the gauge-fixed action (5.6) is the free two-dimensional wave equation

$$\frac{\partial^2 x^\mu}{\partial \tau^2} - \frac{\partial^2 x^\mu}{\partial \sigma^2} = 0 \tag{5.7}$$

There are two types of solution for this wave equation, corresponding to open strings and closed strings. For open strings the solution must satisfy $\acute{x}^\mu = 0$ at

the string end points $\sigma = 0$ and $\sigma = \pi$. This solution is

$$x^\mu(\tau,\sigma) = x_0^\mu + p_0^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma \tag{5.8}$$

where x_0^μ and p_0^μ are the center of mass position and momentum.

For closed strings the solution must satisfy the periodic boundary condition $x^\mu(\tau,\sigma) = x^\mu(\tau,\sigma+\pi)$. The solution can then be decomposed into right-moving waves and left-moving waves,

$$x^\mu(\tau,\sigma) = x_R^\mu(\tau - \sigma) + x_L^\mu(\tau + \sigma) \tag{5.9}$$

$$x_R^\mu = \frac{1}{2} x_0^\mu + \frac{1}{2} p_0^\mu (\tau - \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau-\sigma)} \tag{5.10}$$

$$x_L^\mu = \frac{1}{2} x_0^\mu + \frac{1}{2} p_0^\mu (\tau + \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \tag{5.11}$$

The open string solution (5.8) played a central role in the developments of [8] and in a wider perspective solutions (5.8-5.11) form the basis of the most traditional quantization procedure in bosonic string theory [17]. But there is an inconvenience in this quantization procedure: the Weyl invariance (5.5) is not preserved in the quantized theory. Only in (25+1) space-time dimensions can we construct a consistent quantized bosonic string theory based on the conformal wave equation (5.7) because only in this space-time dimension is the Weyl invariance (5.5) restored in the quantum theory. This situation could in principle be avoided if we could find a way to the wave equation (5.7) that does not require the Weyl invariance of the theory. We now show that we can find this way by treating the string as a constrained system, in much the same way as we did for the relativistic particle. Following this way, we will also be able to show that the basic commutator structure we found for the massless particle can also be encountered in the high energy limit of string theory.

In the transition to the Hamiltonian formalism the Nambu-Goto action (5.1) gives the canonical momentum

$$p_\mu = -T\sqrt{-g}\, g^{0a} \partial_a x_\mu \tag{5.12}$$

and this momentum gives rise to the primary constraints

$$\Phi_0 = \frac{1}{2}(p^2 + T^2 \acute{x}^2) = 0 \tag{5.13}$$

$$\Phi_1 = p.\acute{x} = 0 \tag{5.14}$$

The canonical Hamiltonian corresponding to action (5.1) identically vanishes and so we can construct the first-order Lagrangian density

$$L = p.\dot{x} - \frac{\lambda_0}{2}(p^2 + T^2 \acute{x}^2) - \lambda_1 p.\acute{x} \tag{5.15}$$

where the two-dimensional fields λ_0 and λ_1 play the role of Lagrange multipliers. Solving the equation of motion for p_μ that follows from (5.15) and inserting the result back in it, we obtain the string action

$$S = \int d\tau d\sigma [\frac{1}{2}\lambda_0^{-1}(\dot{x} - \lambda_1 \acute{x})^2 - \frac{1}{2}\lambda_0 T^2 \acute{x}^2] \tag{5.16}$$

Action (5.16) is the higher-dimensional extension of the particle action (3.6). Because the constraints Φ_0 and Φ_1 are first-class, now we have two arbitrary functions, λ_0 and λ_1, at our disposal. Under infinitesimal reparametrizations we have

$$\delta(\partial_a x^\mu) = -\partial_a \epsilon^b \partial_b x^\mu \tag{5.17}$$

and action (5.16) will be reparametrization-invariant if

$$\delta\lambda_0 = (\acute{\epsilon}_1 - \dot{\epsilon}_0)\lambda_0 - \lambda_1\lambda_0\acute{\epsilon}_0 \tag{5.18a}$$

$$\delta\lambda_1 = (\acute{\epsilon}_1 - \dot{\epsilon}_0)\lambda_1 + \dot{\epsilon}_1 - \acute{\epsilon}_0\lambda_1^2 - T^2\lambda_0^2\acute{\epsilon}_0 \tag{5.18b}$$

The string action (5.16) then has the usual reparametrization invariance of the Nambu-Goto action. The classical equation of motion for x^μ that follows from action (5.16) is

$$\frac{\partial}{\partial\tau}(\frac{1}{\lambda_0}\dot{x}^\mu - \frac{\lambda_1}{\lambda_0}\acute{x}^\mu) + \frac{\partial}{\partial\sigma}[-\frac{\lambda_1}{\lambda_0}\dot{x}^\mu + (\frac{\lambda_1^2}{\lambda_0} - \lambda_0 T^2)\acute{x}^\mu] = 0 \tag{5.19}$$

Choosing first $\lambda_0 = 1$, if we want to reach the conformal wave equation, we must satisfy the condition $\lambda_1^2 - T^2 = -1$. This condition means that $\lambda_1 = \pm\sqrt{T^2 - 1}$. Using $\lambda_0 = 1$, the positive value of λ_1 in the first term of (5.19), and the negative value of λ_1 in the second term, equation (5.19) becomes

$$\frac{\partial^2 x^\mu}{\partial\tau^2} - \frac{\partial^2 x^\mu}{\partial\sigma^2} = 0$$

which is identical to the conformal wave equation (5.7), obtained from action (5.4) after imposing the Weyl invariance of the theory.

Returning to the string action (5.16), we now use the reparametrization invariance to impose the gauge $\lambda_1 = 0$. It can be verified that constraints (5.13) and (5.14) are preserved in this gauge. Choosing again $\lambda_0 = 1$, we find that in the energy scale where the string tension has an unitary value, the resulting gauge-fixed action can be written as

$$S = \int d\tau d\sigma (\frac{1}{2}\sqrt{-\det \mathbf{h}}\mathbf{h}^{-1}\mathbf{Trg}) \tag{5.20}$$

where the matrix fields

$$\mathbf{h} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5.21}$$

$$\mathbf{g} = \left[\begin{array}{cc} \partial_0 x.\partial_0 x & \partial_0 x.\partial_1 x \\ \partial_1 x.\partial_0 x & \partial_1 x.\partial_1 x \end{array} \right] \tag{5.22}$$

were introduced. Action (5.20) is in the usual form of a non-commutative field theory action [10]. It can be checked that the conformal wave equation (5.7) directly follows from an action principle based in (5.20).

Returning once more to our starting string action (5.16), we choose again $\lambda_1 = 0$ but now select the energy scale where the string tension vanishes. The string action in this limit becomes

$$S = \frac{1}{2} \int d\tau d\sigma \lambda_0^{-1} \dot{x}^2 \tag{5.23}$$

and to be fully consistent, must be complemented with the constraints

$$\Phi_0 = \frac{1}{2} p^2 = 0 \tag{5.24}$$

$$\Phi_1 = p.\acute{x} = 0 \tag{5.25}$$

which are the $T = 0$ limit of constraints (5.13-5.14). The equation of motion for x^μ that follows from the tensionless string action (5.23) is identical in form to the equation of motion for x^μ that follows from the massless particle action (3.8) (with the total τ-derivative replaced by a partial τ-derivative). We see that in the high energy limit there is a string motion in which each point of the string moves as a massless particle. As a consequence, the string action (5.23) is invariant under the scale transformation (3.11) and under the conformal transformation (3.12). Relaxing again the mass-shell condition (5.24) we find that the tensionless action (5.23) is also invariant under the transformation (3.13), the only difference being that now $x^\mu = x^\mu(\tau, \sigma)$. The commutator structure (3.20-3.22) has then a natural extension in the high energy limit of bosonic string theory. This extension is simply given by

$$[\tilde{p}_\mu(\sigma), \tilde{p}_\nu(\sigma')] = 0 \tag{5.26}$$

$$[\tilde{x}_\mu(\sigma), \tilde{p}_\nu(\sigma')] = \{i\delta_{\mu\nu}(1+\beta)^2 + (1+\beta)[x_\mu, \beta]p_\nu\}\delta(\sigma - \sigma') \tag{5.27}$$

$$[\tilde{x}_\mu(\sigma), \tilde{x}_\nu(\sigma')] = (1+\beta)\{x_\mu[\beta, x_\nu] - x_\nu[\beta, x_\mu]\}\delta(\sigma - \sigma') \tag{5.28}$$

6 Superstrings

We now explicitly check that the bosonic commutators (5.26-5.28) remain invariant in the high energy limit of superstring theory. The Green-Schwarz spacetime supersymmetric version of the Nambu-Goto string can be written as

$$S = -T \int d\tau d\sigma (\sqrt{-g} + L_{WZ}) \tag{6.1}$$

where g is the determinant of the induced two-dimensional metric $g_{ab} = \delta_{\mu\nu} Z_a^\mu Z_b^\nu$ with $Z_a^\mu = \partial_a X^\mu - i\bar{\theta}\Gamma^\mu \partial_a \theta$. L_{WZ} is the Wess-Zumino term

$$L_{WZ} = -i\epsilon^{ab} Z_a . \bar{\theta}\Gamma\partial_b\theta \tag{6.2}$$

which is necessary for the presence of the local fermionic symmetry [17]. This symmetry is important at the classical level, but we will present evidence that it may be discarded in the quantized theory. Switching off the fermions in action (6.1) we recover the original Nambu-Goto action (5.1) for the bosonic string. At this point it should be clear to the reader that we can immediately construct the supersymmetric extension of the bosonic string action (5.16) by simply making the substitution $\partial_a x^\mu \to Z_a^\mu$ and adding the Wess-Zumino term. But let us check this explicitly.

The canonical momenta that follow from action (6.1) are

$$P_\mu = -T(\sqrt{-g} g^{0a} Z_{a\mu} + i\bar{\theta}\Gamma_\mu \dot{\theta}) \tag{6.3}$$

$$\Pi_\alpha = i(P - TZ_1).(\Gamma\theta)_\alpha \tag{6.4}$$

and these momenta give rise to the primary constraints

$$\Phi_0 = \frac{1}{2}(Q^2 + T^2 Z_1^2) = 0 \tag{6.5}$$

$$\Phi_1 = Q.Z_1 = 0 \tag{6.6}$$

$$\Psi_\alpha = \Pi_\alpha - i(P - TZ_1).(\Gamma\theta)_\alpha = 0 \tag{6.7}$$

where we introduced the mechanical momentum

$$Q_\mu = P_\mu + iT\bar{\theta}\Gamma_\mu \acute{\theta} \tag{6.8}$$

The canonical Hamiltonian corresponding to action (6.1) identically vanishes and so Dirac's Hamiltonian density for the Nambu-Goto superstring is

$$H_D = \lambda^a \Phi_a + \bar{\varsigma}\Psi \tag{6.9}$$

where λ^a are two bosonic Lagrange multipliers and $\bar{\varsigma}$ is a fermionic one. The Lagrangian density corresponding to (6.9) is

$$L = P.\dot{X} + \bar{\Pi}\dot{\theta} - \lambda^a \Phi_a - \bar{\varsigma}\Psi \tag{6.10}$$

If we solve the classical equations of motion for P_μ and $\bar{\Pi}_\alpha$ that follow from an action principle based in (6.10), and insert the results back in it, we obtain the superstring action

$$S = \int d\tau d\sigma [\frac{1}{2}\lambda_0^{-1}(Z_0 - \lambda^1 Z_1)^2 - \frac{1}{2}\lambda^0 T^2 Z_1^2 + T L_{WZ}] \tag{6.11}$$

Action (6.11) is the supersymmetric extension of the bosonic string action (5.16). In fact, under the infinitesimal reparametrizations

$$\delta(\partial_a X^\mu) = -\partial_a \epsilon^b \partial_b X^\mu \tag{6.12a}$$

$$\delta(\partial_a \theta_\alpha) = -\partial_a \epsilon^b \partial_b \theta_\alpha \tag{6.12b}$$

the Wess-Zumino term is well-known to be invariant [17], and action (6.11) becomes reparametrization-invariant if λ_0 varies as in equation (5.18a) and λ_1 as in equation (5.18b). We can again impose the gauge $\lambda_0 = 1, \ \ \lambda_1 = 0$ and in the energy scale where the string tension has an unitary value we may write the non-commutative supersymmetric action

$$S = \int d\tau d\sigma (\frac{1}{2}\sqrt{-\det \mathbf{h}}\mathbf{h}^{-1}\mathbf{Tr}\mathbf{g} + L_{WZ}) \tag{6.13}$$

where now

$$\mathbf{g} = \begin{bmatrix} Z_0.Z_0 & Z_0.Z_1 \\ Z_1.Z_0 & Z_1.Z_1 \end{bmatrix} \tag{6.14}$$

It is interesting to discard the Wess-Zumino term in action (6.13). This is equivalent to relaxing the fermionic constraints (6.7) and being off shell in the fermionic variables. Dropping L_{WZ} and varying X_μ in action (6.13) we arrive at the equations

$$\frac{\partial^2 X^\mu}{\partial \tau^2} - \frac{\partial^2 X^\mu}{\partial \sigma^2} = 0 \tag{6.15a}$$

$$\frac{\partial^2 \theta_\alpha}{\partial \tau^2} - \frac{\partial^2 \theta_\alpha}{\partial \sigma^2} = 0 \tag{6.15b}$$

The same set of equations is arrived at by varying θ_α in action (6.13) without the L_{WZ} term. The set (6.15) is the supersymmetric extension of the conformal equation (5.7).

Obviously we may again impose the necessary conditions to arrive at the supersymmetric extension of the tensionless bosonic string action (5.23). This extension is given by the action

$$S = \frac{1}{2}\int d\tau d\sigma \lambda^{-1} Z_0^2 \tag{6.16}$$

complemented with the constraints

$$\Phi_0 = \frac{1}{2}P^2 = 0 \tag{6.17}$$

$$\Phi_1 = P.Z_1 = 0 \tag{6.18}$$

$$\Psi_\alpha = \Pi_\alpha - iP.\Gamma\theta_\alpha = 0 \tag{6.19}$$

From our previous experience with the massless superparticle we note that the bosonic commutators (5.26-5.28) must be preserved in the superstring theory defined by action (6.16).

7 Relativistic membranes

As a final task, we now briefly show that the basic bosonic commutator structure we found for the massless particle and for the tensionless string also exists in the tensionless limit of relativistic membrane theory. For a review of membrane theory see [22].

A relativistic membrane propagating in space-time may be described by a Nambu-Goto-Dirac-type action given by

$$S = -T \int d\tau d^2\sigma \sqrt{-g} \tag{7.1}$$

with

$$g = \det g_{AB} \tag{7.2}$$

$$g_{AB} = \delta_{\mu\nu}\partial_A X^\mu \partial_B X^\nu \tag{7.3}$$

where now $X^\mu = X^\mu(\tau, \sigma_1, \sigma_2)$ and $A = 0, 1, 2$. Action (7.1) gives the canonical momentum

$$P_\mu = -T\sqrt{-g}g^{0A}\partial_A X_\mu \tag{7.4}$$

and this momentum gives rise to the primary constraints

$$\Phi_0 = \frac{1}{2}(P^2 + T^2\tilde{g}) = 0 \tag{7.5}$$

$$\Phi_a = P.\partial_a X = 0 \tag{7.6}$$

where $\tilde{g} = \det(\partial_a X.\partial_b X)$ and $a = 1, 2$. Performing the same manipulations as in the bosonic string, we arrive at the membrane action

$$S = \int d\tau d^2\sigma [\frac{1}{2}\lambda_0^{-1}(\dot{X} - \lambda^a\partial_a X)^2 - \frac{1}{2}\lambda_0 T^2\tilde{g}] \tag{7.7}$$

Action (7.7) is the higher-dimensional extension of the string action (5.16) It is reparametrization invariant for

$$\delta\lambda^0 = (\partial_a\epsilon^a - \partial_0\epsilon^0)\lambda^0 - 2\lambda^0\lambda^a\partial_a\epsilon^0 \tag{7.8a}$$

$$\delta\lambda^a = (\partial_0 + \lambda^b\partial_b)\epsilon^a - \lambda^a(\partial_0 + \lambda^b\partial_b)\epsilon^0 - (\lambda^0)^2 T^2 \tilde{g}\tilde{g}^{ab}\partial_b\epsilon^0 \tag{7.8b}$$

Equations (7.8b) show that there is enough reparametrization freedom to impose the gauge $\lambda_1 = \lambda_2 = 0$. In this gauge we get the membrane action

$$S = \int d\tau d^2\sigma (\frac{1}{2}\lambda_0^{-1}\dot{X}^2 - \frac{1}{2}\lambda_0 T^2 \tilde{g}) \tag{7.9}$$

It is easy to check that action (7.9) preserves the constraints (7.5) and (7.6). Going to the energy region where the membrane tension vanishes, the partially gauge-fixed action (7.9) becomes

$$S = \frac{1}{2}\int d\tau d^2\sigma \lambda_0^{-1}\dot{X}^2 \tag{7.10}$$

complemented with the constraints

$$\Phi_0 = \frac{1}{2}P^2 = 0 \tag{7.11}$$

$$\Phi_a = P.\partial_a X = 0 \tag{7.12}$$

We see that each point of this partially gauge-fixed tensionless membrane moves as a massless particle, and each line moves as a tensionless string. As a consequence of this, action (7.10) is invariant under the scale transformation (3.11), under the conformal transformation (3.12) and also under the transformation (3.13), now with $X^\mu = X^\mu(\tau, \sigma_1, \sigma_2)$. The commutator structure (5.26-5.28) may then be further extended to

$$[\tilde{P}_\mu(\sigma_{1,}\sigma_2), \tilde{P}_\nu(\sigma_1', \sigma_2')] = 0 \tag{7.13}$$

$$[\tilde{X}_\mu(\sigma_1, \sigma_2), \tilde{P}_\nu(\sigma_1', \sigma_2')] = \{i\delta_{\mu\nu}(1+\beta)^2.$$

$$+(1+\beta)[X_\mu, \beta]P_\nu\}\delta(\sigma_1 - \sigma_1')\delta(\sigma_2 - \sigma_2') \tag{7.14}$$

$$[\tilde{X}_\mu(\sigma_1, \sigma_2), \tilde{X}_\nu(\sigma_1', \sigma_2')] = (1+\beta)\{X_\mu[\beta, X_\nu]$$

$$-X_\nu[\beta, X_\mu]\}\delta(\sigma_1 - \sigma_1')\delta(\sigma_2 - \sigma_2') \tag{7.15}$$

The space-time supersymmetric extension of action (7.7) may now be immediately written,

$$S = \int d\tau d^2\sigma [\frac{1}{2}\lambda_0^{-1}(Z_0 - \lambda^a Z_a)^2 - \frac{1}{2}\lambda_0 T^2 \tilde{G} + TL_{WZ}] \tag{7.16}$$

where now $\tilde{G} = \det Z_a.Z_b$, and the Wess-Zumino term is

$$L_{WZ} = -i\epsilon^{ABC}\bar{\theta}\Gamma_{\mu\nu}\partial_A\theta(Z_B^\mu\partial_C X^\nu - \frac{1}{3}\bar{\theta}\Gamma^\mu\partial_B\theta\bar{\theta}\Gamma^\nu\partial_C\theta) \tag{7.17}$$

with $\Gamma_{\mu\nu} = \Gamma_{[\mu}\Gamma_{\nu]}$. It is clear that the supersymmetric extension of the tensionless membrane action (7.10) can be immediately constructed, and its invariance under the membrane extension of the massless superparticle transformation (4.8) will lead to bosonic membrane canonical variables satisfying commutators (7.13-7.15).

8 Concluding remarks

In this work we studied relativistic particle, string and membrane theories as defining field theories containing gravity in (0+1), (1+1) and (2+1) space-time dimensions, respectively. The massless limit of particle theory, and the tensionless limit of string and membrane theories were investigated here using alternative Lagrangian formulations obtained by incorporating the Hamiltonian constraints into the formalism. These limits were interpreted as describing the high energy limits of the corresponding theories. We showed how we may use the ortogonality condition between the relativistic velocity and the relativistic acceleration, a condition which in a certain sense defines the region of applicability of Special Relativity, to induce the appearance of a new invariance of the massless particle action. It was then described how this invariance allows the construction of an off shell extension of the conformal algebra and induces a transition to new space-time coordinates which obey non-vanishing commutation relations. These commutation relations are different from the ones usually encountered in the modern formulations of non-commutative field theory and suggest a dynamical, rather than static, space-time geometry. These bosonic commutation relations remain unaltered for the massless supersymmetric particle action. It was then shown how extensions of these non-vanishing bosonic commutation relations can also be encountered in the tensionless limit of string and membrane theory, both in the bosonic and supersymmetric sectors.

Using special-relativistic concepts only, we showed how we may construct a simple model of a relativistic particle with an internal structure, an idea originally developed by Dirac [23] in 1971. We then proposed that the non-locality introduced by the commutators we found for the massless particle should be interpreted as an indication that at very high energies the massless particle could behave as a gravitational dipole. The presence of the same basic commutator structure in the tensionless limit of string and membrane theories should then be interpreted as evidences that there are possible motions in which these relativistic objects behave as linear and surface distributions of gravitational dipoles, respectively. What role would the concept of a gravitational dipole play in Physics?

A gravitational dipole has a dynamical behavior much different from that of an electric dipole. While opposite electric charges attract themselves, it is seen from Newton's equation for the gravitational force

$$\vec{F} = -\frac{\gamma m_1 m_2}{r^2}\vec{u}_r \tag{8.1}$$

that opposite masses should repel each other. Then, contrary to electric dipoles

which require an external electric field to prevent them from collapsing, gravitational dipoles may exist indefinitely as consequences of their own internal repulsive gravitational forces.

Newton's theory for the gravitational field $\vec{G}$ is contained in the equations

$$\vec{\nabla}.\vec{G} = 4\pi\gamma\rho \tag{8.2a}$$

$$\vec{\nabla} \times \vec{G} = 0 \tag{8.2b}$$

Equations (8.2) are identical in form to Maxwell's equations in the absence of electric currents and magnetic fields

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0} \tag{8.3a}$$

$$\vec{\nabla} \times \vec{E} = 0 \tag{8.3b}$$

It is well-known that equations (8.3) are just a part of a complete set of equations, the Maxwell equations

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0} \tag{8.4a}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{8.4b}$$

$$\vec{\nabla}.\vec{B} = 0 \tag{8.4c}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{8.4d}$$

and that this set of four equations can be naturally written in covariant form with the introduction of a four potential $A^\mu = (\phi, \vec{A})$.

Now, a stationary point mass constitutes a gravitational charge only for observers at rest relative to it or, at most, for observers moving with very small velocities. For fast-moving observers a gravitational charge looks like a gravitational current. Gravitational currents are then necessary ingredients to achieve covariance. It now seems that, to get a covariant Newtonian gravitation, it is only necessary to have a gravitational analogue of the magnetic field $\vec{B}$, a field, say $\vec{M}$, satisfying

$$\vec{\nabla}.\vec{M} = 0 \tag{8.5a}$$

$$\vec{M} = \vec{\nabla} \times \vec{W} \tag{8.5b}$$

for some vector potential $\vec{W}$. Perhaps the gravitational dipoles could be used as sources for the field $\vec{M}$, as magnetic dipoles are sources for the field $\vec{B}$. As the magnetic field $\vec{B}$ is a manifestation of electric charges in motion, the field $\vec{M}$ would be a manifestation of gravitational charges in motion, and this is reflected

in the fact that the gravitational dipole picture can only be constructed in the relativistic theory. It may be interesting to investigate the internal consistency of such a complemented Newtonian gravitational field theory. It would describe attractive and repulsive gravitational interactions, intermediated by spin-one gauge particles, in a flat universe which would be naturally expanding due to the presence of negative matter.

This work is dedicated to the 100^{th} anniversary of Special Relativity (1905-2005).

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